**Math 522 Exam 11 Solutions**

1. For $n = 13$, $a = 2$ is a primitive root, whose index table is below for convenience. Use indices to find all solutions to the congruence $5x^{4^{987}} \equiv x^{11^{654}} \pmod{13}$.

| k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $2^k$ | 2 | 4 | 8 | 3 | 6 | 12 | 11 | 9 | 5 | 10 | 7 | 1 |

We have $ind\ 5x^{4^{987}} \equiv ind\ x^{11^{654}} \pmod{12}$. Using the index properties we simplify as $ind\ 5 + 4^{987}\ ind\ x \equiv 11^{654}\ ind\ x \pmod{12}$. Note that $4^2 \equiv 4 \pmod{12}$, hence $4^{987} \equiv 4$. Note also that $11 \equiv -1$, so $11^{654} \equiv (-1)^{654} \equiv 1 \pmod{12}$. We therefore simplify as $3(ind\ x) \equiv -9 \equiv 3 \pmod{12}$. This has three solutions, $ind\ x \equiv 1, 5, 9 \pmod{12}$, which correspond to $x \equiv 2, 6, 5 \pmod{13}$.

2. Find a primitive root modulo $4394 = 2 \cdot 13^3$.

From the first problem, we know that 2 is a primitive root modulo 13. We calculate $2^{12} \equiv 40 \pmod{13^2}$. Since this isn't 1, we know that 2 is a primitive root modulo $13^2$, and hence also modulo $13^3$. However, since 2 is not odd, it is not a primitive root modulo 4394; instead $2 + 13^3 = 2199$ is.